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M. Sc. (Sem.-IV) Examination September - 2025

Maths : Advanced Linear Algebra - 402

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) There are five questions in this question paper (3) Answer all questions (4) Figure to the right indicates full marks of the questions.	Student's Signature

Q.1. Attempt any Two **[14]**

A. Define Homomorphism and If V and W are vector space over a field F then show that $Hom\,(V, W)$ is also a vector space over F.

B. If V be a finite dimensional vector space over F and $\{v_1, v_2, ..., v_n\}$ be a basis of V.

Let $\hat{v}_i(v_j) = \begin{cases} 0\,;\, i \neq j \\ 1\,;\, i = j \end{cases}$ then show that $\hat{v}_i$ is a linear transformation for all $i = 1, 2, ..., n$ and $\{\hat{v}_1\ \hat{v}_2 ..., \hat{v}_n\}$ is a basis of $\widehat{V}$. Hence prove that $\dim \widehat{V} = \dim V$.

C. If the system of homogeneous linear equation

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}\,x_n = 0 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}\,x_n = 0 \end{matrix}$$

where $a_{ij} \in F$ is of rank r then show that there are $n - r$ linearly independent solution in $F^{(n)}$.

Q.2. Attempt any Two **[14]**

A. Find rank and nullity of the system of equation,

$$x_1 + 2x_2 + x_3 + 2x_4 = 0$$
$$2x_1 + 4x_2 + x_3 + 3x_4 - x_5 = 0$$
$$x_1 + 2x_2 - 2x_3 - x_4 - 3x_5 = 0$$
$$3x_1 + 6x_2 - 2x_3 + x_4 - 5x_5 = 0$$

B. If A is an algebra with a unit element over F then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F.

C. If V is a finite dimensional vector space over F, then for $S, T \in A(V)$ prove the following

a) $r(ST) \leq r(T)$

b) $r(TS) \leq r(T)$

c) $r(ST) = r(TS) = r(T)$ for S is regular in $A(V)$.

Q.3. Attempt any Two **[14]**

A. Define Characteristic Root and if V is a vector space over F and elements $\lambda \in F$ is a characteristic root of $T \in A(V)$ then show that for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$.

B. Let V is a vector space over F and $S,T \in A(V)$ also S is regular then show that T and STS^{-1} have the same minimal polynomial.

C. IF $W \subseteq V$ is invariant under T then prove that

1. T induces a linear transformation $\overline{T}$ on V/W defined by $(v + W)\ \overline{T} = vT + W$
2. If T satisfies the polynomial $q(x) \in F[x]$ then so does $\overline{T}$.
3. If $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and if $p(x)$ is that for T then $p_1(x)|p(x)$.

Q.4. Attempt any Two **[14l**

A. Define similar matrices and if the matrix $A \in F_n$ has all its characteristic roots in F, then prove that there is a matrix $C \in F_n$ such that CAC^{-1} is a triangular matrix.

B. Let V be a finite dimensional vector space over F and if $T \in A(V)$ is nilpotent then show that invariants of T are unique.

C. Let V be a finite dimensional vector space over F, $T \in A(V)$ and $V = V1 \oplus V2$ where V_1 and V_2 are subspaces of V is invariant under T. Let T_1 and T_2 be the linear transformations induced by T on V_1 and V_2 respectively. If the minimal polynomial of T_1 and T_2 over F is $P_1(x)\ P_2(x)$ respectively then show that the minimal polynomial for T over F is the least common multiple of $P_1(x)$ and $P_2(x)$.

Q.5. Attempt any Two **[14]**

A. If $T \in A(V)$ and V is finite dimensional vector space over F has minimal polynomial $p(x) = q(x)^e$; where $q(x)$ is monic irreducible polynomial over F then show that there is a basis for V in which the matrix of T is of the form

$$\begin{pmatrix} C(q(x)^{e_1}) & & & \\ & C(q(x)^{e_2}) & & \\ & & \ddots & \\ & & & C(q(x)^{e_r}) \end{pmatrix}$$ Where $e_1 \geq e_2 \geq \ldots \geq e_r$ and $e = \sum_{i=1}^{r} e_i$.

B. If F is a field of rational numbers then find all possible rational canonical forms and elementary divisors for 8×8 matrices in F_8 having $(x^4 - 4x^3 + 6x^2 - 4x - 7)(x - 3)^2$ as minimal polynomial.

C. Find Jordan canonical form of $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$